NONLINEAR NEAR-RESONANCE OSCILLATIONS OF A GAS IN A TUBE OF VARIABLE CROSS SECTION

A. L. Ni

UDC 534.2:532

The nonlinear near-resonance oscillations of a gas in a tube of variable cross section are investigated; one end of the tube is closed, and the other end is terminated in a piston, the velocity of which, $u_p = \delta a_0 f(t)$, is a periodic function of the time (a_0 is the sound velocity in the undisturbed gas, $\delta \ll 1$). The three-dimensional nature of the basic problem is taken into account in the quasi-one-dimensional approximation. The analysis is carried out in a range of frequencies close to the resonance frequencies of a cylindrical tube of the same length. The influence of viscosity is neglected.

The problem has been investigated previously [1] in a similar setting. Its analysis was based on a method [2, 3] whereby the solution is presented in the form of a power series in a small parameter $\varepsilon = \varepsilon(\delta)$. The problem was solved in the special case of a tube configuration such that the equation describing the gas oscillations was reduced to an ordinary differential equation [2, 3]. An integrodifferential equation was derived to cover a more general situation, and some of its qualitative properties were discussed. To the best of the author's knowledge, quantitative results complementary to [1] have not been obtained to date.

We solve this problem in the present article by the method of [4], the substance of which can be briefly formulated as the representation of the solution by a linear superposition of nonlinear waves of different families. An effective technique has been proposed in [4] for solving the derived nonlinear functional equations, which coincide with the results of [2, 3] in the limit $\varepsilon \to 0$. In [5] the method was generalized to a broader class of problems for $\varepsilon \to 0$, which are reducible to the solution of integrodifferential equations.

Evidently, many standard problems of the oscillations of a gas can be solved by the technique of [4] or some modification thereof. Such problems include subharmonic resonance [6, 7] and oscillations in a tube with allowance for the boundary layer in the approximation of [3]. In the present article we illustrate the capabilities of the method in the example of the problem of near-resonance oscillations of a gas in a variable tube.

By analogy with one-dimensional flows, we refer to the quantities $J^{\pm} = u \pm 2a/(\varkappa - 1)$ as Riemann invariants. We introduce dimensionless dependent and independent (primed) variables by the formula [4].

$$u = \varepsilon a_0 u',\ J^{\pm} = a_0[\varepsilon J^{\pm\prime} \pm 2/(\varkappa - 1)],$$
$$a = a_0(1 + \varepsilon a'),\ t = Tt',\ x = a_0 T x'. \tag{1}$$

Here u is the flow velocity, a is the sound velocity, t is the time, x is the Cartesian coordinate, T is a characteristic time of the flow, a_0 is the unperturbed sound velocity, $\varkappa$ is the adiabatic exponent, and ε is a small parameter characterizing the amplitude of the perturbations. In accordance with the conclusions of [4], the flow can be regarded as isentropic. The equations governing the wave motions of the gas are written in the form

$$(\partial J^+/\partial t)_\xi + \frac{ua}{S} dS/dx = 0,\quad (\partial J^-/\partial t)_\eta - \frac{ua}{S} dS/dx = 0,$$

where the following notation is used for the operators of differentiation along the characteristics $C^{\pm}$:

$$(\partial/\partial t)_\xi = \partial/\partial t + (u + a)\partial/\partial x;\ (\partial/\partial t)_\eta = \partial/\partial t + (u - a)\partial/\partial x;$$

$S = S(x)$ is the cross-sectional area of the tube. As in [1], we set $S = S_0(1 + \varepsilon S')$. The boundary conditions for the investigated problem have the form

$$u(0,\ t) = a_0\delta f(t),\ f(t) = f(t + T),\ u(X,\ t) = 0,$$

Chernogolovka. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 106-111, May-June, 1985. Original article submitted March 21, 1984.

where X is the length of the tube and T is the period of the oscillations of the piston.

Inasmuch as we are investigating near-resonance oscillations, the relation [1] $2X = a_0T(k + \Delta)$ holds, where k is an integer and $\Delta \sim \varepsilon$.

According to [1, 4], for the stated problem to be solvable the accuracy of determination of the flow parameters at the boundaries must be of the order of ε^2. Hence it follows that it is sufficient to calculate the positions of the characteristics within error limits ε, and terms of the order of ε^3 can be neglected in the equations for the invariants.

We substitute Eqs. (1) in the equations of motion. Bearing the foregoing discussion in mind, we obtain the simplified equations

$$(\partial J^+/\partial t)_\xi + \varepsilon(J^+ + J^-)dS/dx/2 = 0,$$
$$(\partial J^-/\partial t)_\eta - \varepsilon(J^+ + J^-)dS/dx/2 = 0; \tag{2}$$
$$C^+:\ dx/dt = 1 + (\varkappa + 1)\varepsilon J^+/4 + (3 - \varkappa)\varepsilon J^-/4, \tag{3}$$
$$C^-:\ dx/dt = -1 + (\varkappa + 1)\varepsilon J^-/4 + (3 - \varkappa)\varepsilon J^+/4;$$
$$J^+(0,\ t) + J^-(0,\ t) = 2\delta f(t)/\varepsilon,\ J^+(n,\ t) + J^-(n,\ t) = 0. \tag{4}$$

We drop the prime from dimensionless variables from now on; $n = (k + \Delta)/2 = O(1)$ is the dimensionless length of the tube. We adopt the oscillation period of the piston as the characteristic time in (1), so that $f(t + 1) = f(t)$.

Below, we follow the analytical procedure of [4]. We integrate (2) and (3) along the corresponding characteristics. It is evident from (2) and (3) that integration along C^+ (C^-) the invariants J^+(J^-) on the right-hand sides of the equations can be taken as constants equal to their values at the initial points of the flow. The remaining integrals, on the other hand, must be computed along the characteristics of the linearized equations:

$$t = x + \xi,\ t = -x + \eta + n.$$

We always interpret the characteristic variables ξ and η as the times of departure of the characteristics C^+(C^-) from the left (right) boundary.

The integration of (3) yields

$$C^+:\ x = \left[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\right](t - \xi) + \frac{3-\varkappa}{8}\varepsilon \int\limits_{\xi-n}^{2t-\xi-n} J^-(\eta)\,d\eta,$$

$$C^-:\ x = n - \left[1 - \frac{\varkappa+1}{4}\varepsilon J^-(\eta)\right](t - \eta) + \frac{3-\varkappa}{8}\varepsilon \int\limits_{\eta-n}^{2t-\eta-n} J^+(\xi)\,d\xi.$$

From this result and from (4) we determine the time t_2 at which the characteristic C^+, emanating from the piston at time ξ_0, returns to it (Fig. 1):

$$t_2 = \xi_0 + 2n\left[1 - \frac{\varkappa+1}{4}\varepsilon J^+(\xi_0)\right] - $$
$$- \frac{3-\varkappa}{8}\varepsilon \int\limits_{\xi_0-n}^{t_1} J^-(\eta)\,d\eta + \frac{3-\varkappa}{3}\varepsilon \int\limits_{\xi_0}^{t_2} J^+(\xi)\,d\xi. \tag{5}$$

Inasmuch as we are considering periodic solutions, the following equations are valid:

$$\int\limits_{\xi_0-n}^{t_1} J^-(\eta)\,d\eta = -\int\limits_{\xi_0}^{t_2} J^+(\xi)\,d\xi = -\int\limits_0^k J^+(\xi)\,d\xi = I_0 = \text{const}, \tag{6}$$

and Eq. (5) coincides with the corresponding equation of one-dimensional gas dynamics [4].

According to the foregoing, all the equations pertaining to the integration of (3) are valid with error $O(\varepsilon)$. Similarly, in the integration of (2) below, the equations are valid with error $O(\varepsilon^2)$ Denoting $dS/dx = \Phi(x)$, we have

$$J^-(t_1, n) = J^-(t_1) = -J^+(t_1, n) =$$

$$= -J^+(\xi_0) + \frac{\varepsilon}{2} J^+(\xi_0) \int_0^n \Phi(x)\,dx + \frac{\varepsilon}{4} \int_{\xi_0 - n}^{t_1} \Phi\left(\frac{\eta - \xi_0 + n}{2}\right) J^-(\eta)\,d\eta,$$

$$J^+(t_2, 0) = J^+(t_2) = -J^-(t_2, 0) + \frac{2\delta}{\varepsilon} f(t_2) = -J^-(t_1) -$$

$$-\varepsilon \frac{J^-(t_1)}{2} \int_0^n \Phi(x)\,dx - \frac{\varepsilon}{4} \int_{\xi_0}^{t_2} \Phi\left(\frac{t_1 + n - \xi}{2}\right) J^+(\xi)\,d\xi + \frac{2\delta}{\varepsilon} f(t_2),$$

$$J^+(t_2) = -\frac{\varepsilon}{2} J^+(\xi_0) \int_0^n \Phi(x)\,dx - \frac{\varepsilon}{4} \int_{\xi_0 - n}^{t_1} \Phi\left(\frac{\eta - \xi_0 + n}{2}\right) J^-(\eta)\,d\eta - \qquad (7)$$

$$-\frac{\varepsilon}{2} J^-(t_1) \int_0^n \Phi(x)\,dx - \frac{\varepsilon}{4} \int_{\xi_0}^{t_2} \Phi\left(\frac{t_1 + n - \xi}{2}\right) J^+(\xi)\,d\xi + \frac{2\delta}{\varepsilon} f(t_2) + J^+(\xi_0),$$

$$J^+(t_2) = J^+(\xi_0) - \frac{\varepsilon}{4} \int_{t_2 - 2n}^{t_2} J^+(\xi)\left[\Phi\left(\frac{t_2 - \xi}{2}\right) - \Phi\left(\frac{\xi - t_2 + 2n}{2}\right)\right] d\xi.$$

Relations (5) and (7) form a closed system of equations, which must be satisfied by the solution of the stated problem. The constant I_0 in (6), as will be evident below, is determined by the initial conditions of the problem.

As far we have ignored the problem of shock waves, which can occur in a flow field. It is readily shown that their position is determined with the required accuracy by the area rule [4] in the domain where the solution is multiple-valued. These areas can be calculated from the profiles transformed in accordance with the laws of one-dimensional gas dynamics [4], i.e., in the determination of the shock positions, as in the determination of the characteristics (5), effects associated with the variation of the tube cross section are inconsequential.

Thus, the flow parameters in our situation and in the case of one-dimensional flows, as is evident from (5) and (6), differ by an amount of the order of ε^2 at any point of the flow. The same remark applied to the shock wave velocities.

Substituting (5) in (7), expanding the resulting equation in a Taylor series, and taking into account the periodicity of the required solution, we arrive at an integrodifferential equation [1] (we omit these simple computations, which are analogous to those in [4, 5], noting only that the result implies the relation $\delta = \varepsilon^2$). The solution of this equation poses a complex problem, whereas the system (5), (7) can be solved according to the scheme of [4].

On the interval [0, 1] we specify a function $J_0(\xi)$ satisfying the condition $J_0(0) = J_0(1)$. We use the transformation (5) to reduce the interval [0, 1] to the interval [t(0), t(1)], which also has unit length. We calculate the new values of $J_1(\xi)$ at the corresponding points according to Eq. (7) without regard for the integral term, i.e., according to the one-dimensional gasdynamical equation [4]. If domains of multiple value occur in the distribution $J_1(\xi)$, we introduce second-order discontinuities therein according to the area rule. Regarding $J_1(\xi)$ as a periodic function, we calculate the integral term in Eq. (7) from this function and add it to $J_1(\xi)$. It is seen at once that this new distribution $J_2(\xi)$ is related to $J_0(\xi)$ by Eq. (7), correct to within ε. We determine $J_2(\xi)$ on [0, 1] from the periodicity condition and repeat the indicated procedure until the result stabilizes. We note that $\int_0^1 J_2(\xi)\,d\xi = \int_0^1 J_0(\xi)\,d\xi = -I_0$, and this quantity is preserved in the course of the iterations. In Eq. (6), therefore, $I_0 = -k \int_0^1 J_0(\xi)\,d\xi$.

We have used the above-described algorithm to carry out computations for various tube profiles $\Phi(x)$, laws f(t) governing the motion of the piston, and values of the small parameters Δ and δ.

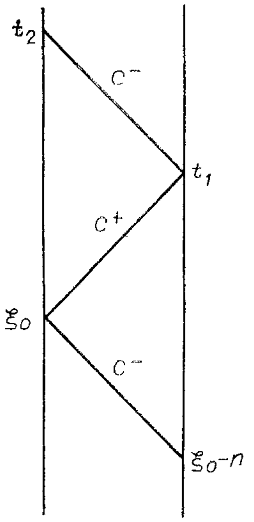

Fig. 1

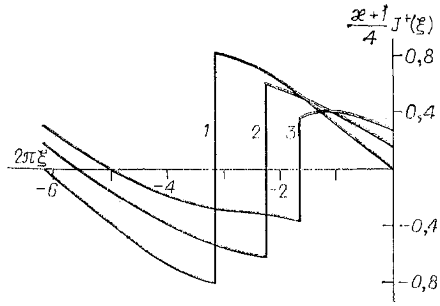

Fig. 2

Figure 2 shows the results of the computations, in which it was assumed that $f(t) = [2/(\varkappa + 1)]\sin 2\pi t$, $n = 1/2$, $\varepsilon = 2 \cdot 10^{-2}$, $\Phi = (2\pi)^2 \alpha\xi$, $I_0 = 0$, $\varkappa = 1.4$. Curves 1-3 correspond to the values of $\alpha = 0, 1, 2$. The vertical segments in the figure indicate shock waves. Curve 1 corresponds to resonance oscillations in a cylindrical tube. It is seen that the amplitude of the periodic oscillations decreases as the opening of the tube is increased. It is conceivable that channels exist with a configuration such that the amplitude of the periodic resonance oscillations in them exceeds the amplitude of the oscillations in a cylindrical tube. However, this kind of situation was not recorded in computations with different functions $\Phi(\xi)$.

All the examples contained in [1-3, 6, 7] pertain to periodic oscillations. Problems of the same class have been solved up to now by the method of [4, 5]. In reality, the domain of validity of the latter is much broader. Essentially, as noted in [4, 5], it is the method of characteristics, which makes it possible, given certain additional assumptions regarding the sought-after solutions, to calculate the positions of the characteristics, the values of the "invariants" transported by the characteristics, and the positions of the shock waves generated from the abrupt intersection of characteristics of one family at finite distances.

As an example illustrating the possibilities of the method we solve the problem of oscillations in a tube with a fairly large opening. We set $\alpha = 3$ in the preceding expression for Φ. Our computations show that the flow in this case becomes quasiperiodic after large times. Superimposed on the solution with unit period is a "long" modulation with a period roughly equal to 60. Oscillograms of $J^0(\xi) = [(\varkappa + 1)/4]J^+(\xi)$ at the left boundary at the times T_0, $T_0 + 20$, $T_0 + 40$, and $T_0 + 60$ are plotted in Fig. 3, in which the indicated times are designated by the numerals 1-4 (the curves are reduced to the interval $[-1, 0]$ with respect to periodicity, and T_0 is sufficiently large that the initial conditions are "forgotten"). It is seen that in a flow containing one shock wave (curve 1) a second shock is generated (curve 2), then the two shocks merge (curve 3), and the flow pattern (curve 4) returns to the original state (curve 1); curves 1 and 4 are indistinguishable in the scale of Fig. 3, and so curve 4 is represented by a dashed line. The only modification that has to be introduced in the above-described algorithm for the given situation is that the solution is not continued onto the interval [0, 1] in the course of the iterations. As for the integral involved in (7), it can be computed, as before, by virtue of the quasiperiodicity of the solution. We denote by ξ_{s1} (ξ_{s2}) the arrival times of the first (second) shock at the piston, and by N an integer such that $N - 1 < \xi_{s1} \leqslant N$. The quasiperiodicity of the process is readily discerned from Fig. 4, in which the solid curve represents a broken line connecting the points $[2\pi(\xi_{s1} - N), (N - N_0)/60]$, and N_0 is a sufficiently large number to allow stabilization of the indicated regime; this number depends on the intial conditions. Calculations show that the initial conditions do not affect the final solution.

The resulting flow pattern is explained by the interference of waves having a period that is a multiple of the period of the driving "force" with the proper solution for a variable channel. With an increase in α the role of such waves increases. A further increase in α makes the flow pattern even more complicated.

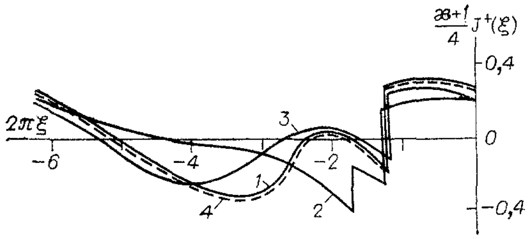

Fig. 3

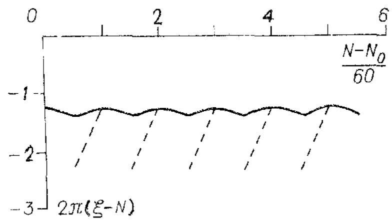

Fig. 4

A detailed analysis of the causes underlying the onset of the "long" modulation would be of independent interest and is not carried out in the present study.

The author is grateful to V. E. Fortov and S. I. Anisimov for their interest and encouragement.

LITERATURE CITED

1. J. J. Keller, "Nonlinear acoustic resonances in shock tubes with varying cross-sectional area," Z. Angew. Math. Phys., 28, No. 1, 107 (1977).
2. L. P. Gor'kov, "Nonlinear acoustic oscillations of a gas column in a closed tube," Inzh. Zh., 3, No. 2 (1963).
3. W. Chester, "Resonant oscillations in closed tubes," J. Fluid Mech., 18, Part 11 (1964).
4. A. N. Kraiko and A. L. Ni, "Nonlinear acoustics approximation in problems of gas oscillations in tubes," Prikl. Mat. Mekh., 44, No. 1 (1980).
5. A. L. Ni, "Nonlinear resonant oscillations of a gas in a tube under the action of a periodically varying pressure," Prikl. Mat. Mekh., 47, No. 4 (1983).
6. Sh. U. Galiev, N. A. Il'gamov and A. V. Sadykov, "Periodic shock waves in a gas," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 2 (1970).
7. M. P. Mortell and B. R. Seymour, "A finite-rate theory of quadratic resonance in a closed tube," J. Fluid Mech., 112, 411 (1981).

INTERACTION OF AIR SHOCK WAVES WITH POROUS COMPRESSIBLE MATERIALS

L. G. Gvozdeva, Yu. M. Faresov,
and V. P. Fokeev

UDC 532.593:532.529

Experimental studies of the process of interaction of air shock waves with porous compressible materials as a polyurethane foam and formed plastics have shown that such interaction has a number of unique features. Thus, it was shown in [1] that the maximum pressure on a wall beneath a layer of polyurethane foam can significantly exceed the value of pressure attained in normal reflection of a shock wave from a rigid wall. It was proposed in [1] that this effect could be explained by the solid phase being set in motion behind the entering shock wave. Intensification of an oblique shock wave upon incidence on a layer of porous compressible material was analyzed in [2]. Interaction of an air shock wave with a porous screen of polyurethane foam was studied in [3], where a significant reduction in peak pressure on the wall was recorded in the presence of an air gap between the wall and screen. The process observed in [3] was described theoretically in [4] by a computation technique first developed for gas dynamic flows with solid particles. Below we will present results of experimental studies of the interaction of a steady-state shock wave with a wall covered by layers of porous compressible material of various thicknesses.

The materials used for the experiment were PPU-3M-1 polyurethane foam and PKhV-1 foamed plastic; the densities of these materials are approximately the same (33 and 50 kg/m^3 respectively), while the rigidity of the foamed plastic is significantly higher. The loading for failure of this plastic is $(4-7)\cdot 10^5$ N/m^2 [5].

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 111-115, May-June, 1985. Original article submitted April 9, 1984.

0021-8944/85/2603-0401$09.50